

The Higher Schur-Multiplier of Certain Classes of Groups ^{*}

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Abstract

The paper is devoted to calculating the higher Schur-multiplier of certain classes of groups with respect to the variety of nilpotent groups. Our results somehow generalize the works of M.R.R. Moghaddam (1979), and N.D. Gupta and M.R.R. Moghaddam (1993).

AMS Mathematical Subject Classification: 20E10, 20F12, 20F18.

Keywords: Schur-multiplier, variety of groups, nilpotent product.

^{*}This research was in part supported by a grant from IPM, Iran.

1 Introduction

Let \mathcal{N}_c be the variety of nilpotent groups of class at most c ($c \geq 1$) and G be an arbitrary group with a free presentation

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1,$$

where F is a free group of countable rank and $R = \ker \pi$. Then the Baer-invariant of G with respect to the variety of nilpotent groups, \mathcal{N}_c , is defined to be

$$\mathcal{N}_c M(G) = R \cap \gamma_{c+1}(F) / [R, {}_c F],$$

where $\gamma_{c+1}(F)$ is the $(c + 1)$ st-term of the lower central series of F and $[R, {}_c F]$ denotes the commutator subgroup $[R, \underbrace{F, F, \dots, F}_{c\text{-times}}]$. One may check that $\mathcal{N}_c M(G)$ is abelian and independent of the choice of the free presentation of G (see [1, 4, 5]). Determining these Baer-invariants of a given group is known to be very useful for classification of groups into isologism classes (see [2, 4]). In 1992, Gupta and Moghaddam [3] calculated the Baer-invariants of nilpotent dihedral groups of class c ($c > 2$) with respect to the variety of nilpotent groups. In this paper we calculate the Baer-invariants of the n^{th} -nilpotent product of two cyclic groups for $n = 2, 3, 4$, under some conditions. Our results generalize the works of Moghaddam [5] and Gupta and Moghaddam [3].

2 Preliminaries and Some Technical Lemmas

Let $G = G_2 \star G_1$ be the free product of two arbitrary groups G_1 and G_2 . Then the n^{th} -nilpotent product of G_1 and G_2 is defined as follows (see also [7]):

$$G_2 \overset{n}{\star} G_1 = \frac{G}{[G_2, G_1, {}_{(n-1)}G]} \text{ for } n > 1.$$

Let $1 \longrightarrow R_i \longrightarrow F_i \xrightarrow{\pi_i} G_i \longrightarrow 1$ be a free presentation for G_i ($i = 1, 2$), where F_1 and F_2 are free groups, and $R_i = \ker \pi_i$. We then obtain the following free presentation for the n^{th} -nilpotent product $G_2 \star G_1$ as follows:

$$1 \longrightarrow R \longrightarrow F \longrightarrow G_2 \star G_1 \longrightarrow 1,$$

where $F = F_2 \star F_1$ and

$$R = \langle R_2, R_1, [F_2, F_1, {}_{(n-1)}F] \rangle^F = R_2 R_1 \prod_{i=1}^2 [R_i, F][F_2, F_1, {}_{(n-1)}F].$$

Now, in the above set up, assume that

$$Z_r = \langle x | x^r = 1 \rangle, \quad Z_s = \langle y | y^s = 1 \rangle$$

be two cyclic groups of orders r and s , respectively, and consider the free presentations for Z_r , Z_s and $Z_r \stackrel{n}{\star} Z_s$, when $F_1 = \langle x \rangle$, $F_2 = \langle y \rangle$, $R_1 = \langle x^r \rangle^{F_1}$, and $R_2 = \langle y^s \rangle^{F_2}$. Clearly,

$$Z_r \stackrel{n}{\star} Z_s = \langle x, y | x^r, y^s, \gamma_{n+1}(F) \rangle,$$

which is denoted by $G_{(r,s,n)}$. Now put $S = \langle R_1, R_2 \rangle^F$, and $\rho_{c+1}(S) = [S, {}_c F]$ for $c \geq 0$. Then

$$S = \rho_1(S) \supset \rho_2(S) \supset \rho_3(S) \dots$$

is a central series in the free group F . One notes that

$$S = \langle R_1, R_2 \rangle^F = R_1 R_2 \prod_{i=1}^2 [R_i, F]$$

and hence in this case $R = S \gamma_{n+1}(F)$.

In this paper we determine the Baer-invariants of the group $G_{(r,s,n)}$ with respect to the variety of nilpotent groups, in different cases, when $c \geq n$.

We keep all the notations throughout the rest of the paper. The following technical lemma is vital in our investigation.

Lemma 2.1. Let F be a free group and freely generated by $\{x, y\}$. Then the following congruence holds, for all positive integers, $c \geq 3$, $r \geq 4$, and $a_i \in \{x, y\}$, modulo $\gamma_{c+5}(F)$;

$$\begin{aligned}
& [x^r, y, a_1, \dots, a_{c-1}] \equiv [x, y, a_1, \dots, a_{c-1}]^r [x, y, x, a_1, \dots, a_{c-1}]^{\binom{r}{2}} \\
& [x, y, a_1, [x, y], a_2, \dots, a_{c-1}]^{\binom{r}{2}} [x, y, a_1, a_2, [x, y, a_1], a_3, \dots, a_{c-1}]^{\binom{r}{2}} \\
& [x, y, x, a_1, [x, y], a_2, \dots, a_{c-1}]^{\binom{r+1}{3}} [x, y, x, a_1, [x, y, x], a_2, \dots, a_{c-1}]^{\binom{r}{3} + \binom{r+1}{3}} \\
& [x, y, x, x, a_1, \dots, a_{c-1}]^{\binom{r}{3}} [x, y, x, [x, y], a_1, \dots, a_{c-1}]^{\binom{r}{3}} \\
& [x, y, x, x, x, a_1, \dots, a_{c-1}]^{\binom{r}{4}},
\end{aligned}$$

where $\binom{r}{k}$ denotes $r!/k!(r-k)!$.

Proof. We use double inductions, and so we only need to show the validity of the congruence for $c = 3$, since the induction on c completes the proof. Hence assume $c = 3$, the proof goes by induction on r ($r \geq 4$). By expanding the left hand side for $r = 4$ and working modulo $\gamma_8(F)$, we obtain the formula rather easily. Now assume the result holds for $r - 1$ ($r \geq 5$), we obtain the following congruence, modulo $\gamma_8(F)$

$$\begin{aligned}
& [x^{r-1}, y, a_1, a_2] \equiv [[x^{r-1}, y][x^{r-1}, y, x][x, y], a_1, a_2] \\
& \equiv [x^{r-1}, y, a_1, a_2][x^{r-1}, y, a_1, a_2, [x, y, a_1]] \\
& [x^{r-1}, y, a_1, [x, y], a_2][x^{r-1}, y, a_1, [x, y, a_1]^{r-1}, a_2] \\
& [x^{r-1}, y, x, a_1, a_2][x^{r-1}, y, x, a_1, [x, y], a_2][x, y, a_1, a_2]
\end{aligned}$$

Clearly, modulo $\gamma_8(F)$

$$[x^{r-1}, y, a_1, a_2, [x, y, a_1]] \equiv [x, y, a_1, a_2, [x, y, a_1]]^{r-1}$$

$$\begin{aligned}
[x^{r-1}, y, a_1, [x, y], a_2] &\equiv [x, y, a_1, [x, y], a_2]^{r-1} \\
[x^{r-1}, y, a_1, [x, y, x]^{r-1}, a_2] &\equiv [x, y, a_1, [x, y, x], a_2]^{(r-1)^2} \\
[x^{r-1}, y, x, a_1, [x, y], a_2] &\equiv [x, y, x, a_1, [x, y], a_2]^{r-1}
\end{aligned}$$

Now, using the induction formula for $r - 1$ to the following commutators

$$[x^{r-1}, y, a_1, a_2] \text{ and } [x^{r-1}, y, x, a_1, a_2]$$

we obtain the desired formula. \square

Now we are able to prove our key result, which shortens the proof of the main theorems in the next section.

Proposition 2.2. Let Z_r and Z_s be two cyclic groups of orders r and s , respectively. Let $F_1/R_1 \cong Z_r$, $F_2/R_2 \cong Z_s$, $F = F_1 \star F_2$, $S = \langle R_1, R_2 \rangle^F$ and $\rho_{c+1}(S) = [S, {}_{c+1}F]$ ($c \geq 1$). Then

- (i) $\gamma_{c+2}(F)\rho_{c+1}(S)/\gamma_{c+2}(F) = dZ \oplus \dots \oplus dZ$ ($r(c+1)$ copies), for all non-negative integers r and s , and $d = (r, s)$.
- (ii) $\gamma_{c+j}(F)\rho_{c+1}(S)/\gamma_{c+j}(F) = dZ \oplus \dots \oplus dZ$ ($\sum_{i=1}^{j-1} r(c+i)$ copies), where $j = 3, 4, 5$ and for $j = 3$, then r and s must be odd, and for $j = 4$ or 5 , then r and s should not be divisible by 2 and 3. Also $r(c+i)$ is the number of basic commutators of weight $c+i$ on two letters.

Proof. (i) Modulo $\gamma_{c+2}(F)$, and using induction argument the following congruences hold:

$$[x^r, a_1, \dots, a_c] \equiv [x, a_1, \dots, a_c]^r ;$$

$$[y^s, a_1, \dots, a_c] \equiv [y, a_1, \dots, a_c]^s ,$$

for all $a_1, \dots, a_c \in F$. Clearly these commutators are elements of $\rho_{c+1}(S)$. We note that $\gamma_{c+2}(F)\rho_{c+1}(S)/\gamma_{c+2}(F)$ is generated by all the r^{th} and s^{th} -powers, and hence $(r, s) = d^{th}$ -powers of the basic commutators of weight

$c + 1$. Thus the property of being abelian gives the result.

(ii) If $j = 3$, then we have the following exact sequence

$$1 \longrightarrow \frac{\gamma_{c+3}(F)\rho_{c+2}(S)}{\gamma_{c+3}(F)} \longrightarrow \frac{\gamma_{c+3}(F)\rho_{c+1}(S)}{\gamma_{c+3}(F)} \longrightarrow \frac{\gamma_{c+3}(F)\rho_{c+1}(S)}{\gamma_{c+3}(F)\rho_{c+2}(S)} \longrightarrow 1.$$

By part (i), $\gamma_{c+3}(F)\rho_{c+2}(S)/\gamma_{c+3}(F)$ is an abelian group generated by all the d^{th} -powers of the basic commutators of weight $c + 2$ on two letters. Now we have the following congruence:

$$[x^r, a_1, \dots, a_c] \equiv [x, a_1, \dots, a_c]^r [x, a_1, x, a_2, \dots, a_c]^{\binom{r}{2}} \pmod{\gamma_{c+3}(F)}.$$

Since r is odd, we obtain

$$[x, a_1, x, a_2, \dots, a_c]^{\binom{r}{2}} \equiv [x^r, a_1, x, a_2, \dots, a_c]^{\frac{r-1}{2}} \pmod{\gamma_{c+3}(F)}.$$

which is in $\rho_{c+2}(S)$. Hence modulo $\gamma_{c+3}(F)\rho_{c+2}(S)$,

$$[x^r, a_1, \dots, a_c] \equiv [x, a_1, \dots, a_c]^r,$$

$$[y^s, a_1, \dots, a_c] \equiv [y, a_1, \dots, a_c]^s.$$

Therefore $\gamma_{c+3}(F)\rho_{c+1}(S)/\gamma_{c+3}(F)\rho_{c+2}(S)$ is freely generated by all the d^{th} -powers of the basic commutators of weight $c + 1$. Hence the above split extension gives the result for $j = 3$.

If $j = 4$, then we have the following exact sequences:

$$1 \rightarrow E = \frac{\gamma_{c+4}(F)\rho_{c+2}(S)}{\gamma_{c+4}(F)\rho_{c+3}(S)} \rightarrow B = \frac{\gamma_{c+4}(F)\rho_{c+1}(S)}{\gamma_{c+4}(F)\rho_{c+3}(S)} \rightarrow D = \frac{\gamma_{c+4}(F)\rho_{c+1}(S)}{\gamma_{c+4}(F)\rho_{c+2}(S)} \rightarrow 1,$$

$$1 \rightarrow C = \frac{\gamma_{c+4}(F)\rho_{c+3}(S)}{\gamma_{c+4}(F)} \rightarrow A = \frac{\gamma_{c+4}(F)\rho_{c+1}(S)}{\gamma_{c+4}(F)} \rightarrow B = \frac{\gamma_{c+4}(F)\rho_{c+1}(S)}{\gamma_{c+4}(F)\rho_{c+3}(S)} \rightarrow 1,$$

which are also split extensions. Using the pro of of part (i) and the case $j = 3$, we obtain

$$E \cong dZ \oplus \dots \oplus dZ \quad (r(c+2) - \text{copies}),$$

$$C \cong dZ \oplus \dots \oplus dZ \quad (r(c+3) - \text{copies}).$$

Now for every generator $[x^r, y, a_1, \dots, a_{c-1}]$ or $[y^s, x, a_1, \dots, a_{c-1}]$ of $\rho_{c+1}(S)$ and using Lemma 2.1 we have, modulo $\gamma_{c+4}(F)$,

$$[x^r, y, a_1, \dots, a_{c-1}] \equiv [x, y, a_1, \dots, a_{c-1}]^r [x, y, x, a_1, \dots, a_{c-1}]^{\binom{r}{2}}$$

$$[x, y, a_1, [x, y], a_2, \dots, a_{c-1}]^{\binom{r}{2}} [x, y, x, x, a_1, \dots, a_{c-1}]^{\binom{r}{3}}.$$

Since 2 and 3 do not divide r , it implies that

$$[x^r, y, a_1, \dots, a_{c-1}] \equiv [x, y, a_1, \dots, a_{c-1}]^r \pmod{\gamma_{c+4}(F)\rho_{c+2}(S)}.$$

Similarly

$$[y^s, x, a_1, \dots, a_{c-1}] \equiv [y, x, a_1, \dots, a_{c-1}]^s \pmod{\gamma_{c+4}(F)\rho_{c+2}(S)}.$$

Thus

$$D \cong dZ \oplus \dots \oplus dZ \quad (r(c+1) - \text{copies}).$$

Now the above exact sequences give the result.

Finally, for $j = 5$ we consider the following split exact sequences

$$1 \rightarrow E_i = \frac{\gamma_{c+5}(F)\rho_{c+i}(S)}{\gamma_{c+5}(F)\rho_{c+i+1}(S)} \rightarrow B_i = \frac{\gamma_{c+5}(F)\rho_{c+1}(S)}{\gamma_{c+5}(F)\rho_{c+i+1}(S)}$$

$$\rightarrow D_i = \frac{\gamma_{c+5}(F)\rho_{c+1}(S)}{\gamma_{c+5}(F)\rho_{c+i}(S)} \rightarrow 1, \quad \text{for } i = 2 \text{ and } 3.$$

$$1 \rightarrow C = \frac{\gamma_{c+5}(F)\rho_{c+4}(S)}{\gamma_{c+5}(F)} \rightarrow A = \frac{\gamma_{c+5}(F)\rho_{c+1}(S)}{\gamma_{c+5}(F)} \rightarrow B_2 = \frac{\gamma_{c+5}(F)\rho_{c+1}(S)}{\gamma_{c+5}(F)\rho_{c+4}(S)} \rightarrow 1.$$

Following the previous procedure we conclude that

$$D_2 \cong dZ \oplus \dots \oplus dZ \quad (r(c+1) - \text{copies}),$$

$$E_i \cong dZ \oplus \dots \oplus dZ \quad (r(c+2) - \text{copies})(i = 2, 3),$$

$$C \cong dZ \oplus \dots \oplus dZ \quad (r(c+4) - \text{copies}).$$

Then the above exact sequences guarantee the result. \square

3 The Main Results

Using the results in the previous section we are able to calculate the higher Schur- multipliers $\mathcal{N}_c M(G)$, with respect to the variety of nilpotent groups of class at most c ($c \geq n$). By the above notations,

$$\begin{aligned} \mathcal{N}_c M(G_{(r,s,n)}) &= \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]} = \frac{S\gamma_{n+1}(F) \cap \gamma_{c+1}(F)}{[S\gamma_{n+1}(F), {}_c F]} \\ &= \frac{\gamma_{c+1}(F)}{\gamma_{c+n+1}(F)\rho_{c+1}(S)} \cong \frac{\gamma_{c+1}(F)/\gamma_{c+n+1}(F)}{\gamma_{c+n+1}(F)\rho_{c+1}(S)/\gamma_{c+n+1}(F)}. \end{aligned}$$

Clearly for $c \geq n$, $\gamma_{c+1}(F)/\gamma_{c+n+1}(F)$ is free abelian group of rank $\sum_{i=1}^n r(c+i)$, where $r(c+i)$ denotes the rank of the lower central factor group $\gamma_{c+i}(F)/\gamma_{c+i+1}(F)$.

One notes that the main problem is to determine the structure of the factor group $\gamma_{c+n+1}(F)\rho_{c+1}(S)/\gamma_{c+n+1}(F)$. Now, if $n = 1$, then by Proposition 2.2 (i) we obtain the following theorem.

Theorem 3.1. Let r and s be arbitrary positive integers. Then for any $c \geq 1$ and $(r, s) = d$,

$$\mathcal{N}_c M(G_{(r,s,1)}) \cong Z_d \oplus \dots \oplus Z_d \quad (r(c+1) - \text{copies}).$$

Remark 3.2. The above theorem gives the complete structure of $\mathcal{N}_c M(Z_r \times Z_s)$ for all $c \geq 1$. This generalizes Theorem 3.3 of Moghaddam [5].

The first two authors [6] in 1997 presented an explicit formula for the higher Schur-multiplier of a finite abelian group with respect to the variety of nilpotent groups of class at most $c \geq 1$, \mathcal{N}_c , as follows:

Theorem 3.3. (M.R.R. Moghaddam and B. Mashayekhy (1997)) Let $G = Z_{n_1} \oplus Z_{n_2} \oplus \dots \oplus Z_{n_k}$ be a finite abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$. Then, for all $c \geq 1$, the higher Schur-multiplier of G is

$$\mathcal{N}_c M(G) \cong Z_{n_2}^{(b_2)} \oplus Z_{n_3}^{(b_3-b_2)} \oplus \dots \oplus Z_{n_k}^{(b_k-b_{k-1})},$$

where b_i is the number of basic commutators of weight $c+1$ on i letters, and $Z_n^{(m)}$ denotes the direct sum of m copies of the cyclic group Z_n .

Proof. See Theorem 2.4 of [6].

Theorem 3.4. If $(r, s) = 1$, then for any $n \geq 1$ and $c \geq 1$,

$$\mathcal{N}_c M(G_{(r,s,n)}) = 1.$$

Proof. Using the commutator identities and induction argument we obtain the following congruences, modulo $\gamma_{n+1}(F)$, where F is the free group on $\{x, y\}$.

$$[a_1, \dots, a_{i-1}, x^r, a_{i+1}, \dots, a_n] \equiv [a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n]^r,$$

$$[a_1, \dots, a_{i-1}, y^s, a_{i+1}, \dots, a_n] \equiv [a_1, \dots, a_{i-1}, y, a_{i+1}, \dots, a_n]^s,$$

for all $a_1, \dots, a_n \in \{x, y\}$. Hence all the r^{th} and s^{th} -powers of the basic commutators of weight n are trivial in the following

$$G_{(r,s,n)} \cong, x, y | x^r, y^s, \gamma_{n+1}(F) > .$$

Hence $G_{(r,s,n)} \cong, x, y | x^r, y^s, \gamma_n(F) > .$ Using the above procedure for the commutators of weight less than n , and after finite number of steps we deduce that

$$G_{(r,s,n)} \cong, x, y | x^r, y^s, \gamma_2(F) > \cong Z_r \times Z_s.$$

Now either of Theorems 3.3 or 3.1 gives the result. \square

Using the notations at the beginning of this section and Proposition 2.2, we can prove the following theorem.

Theorem 3.5.

(i) For any odd integers r and s , and all $c \geq 2$

$$\mathcal{N}_c M(G_{(r,s,2)}) \cong Z_d \oplus \dots \oplus Z_d \quad \left(\sum_{i=1}^2 r(c+i) - \text{copies} \right).$$

(ii) For all non-negative integers r and s , which are not divisible by 2 and 3, then

$$\mathcal{N}_c M(G_{(r,s,3)}) \cong Z_d \oplus \dots \oplus Z_d \quad \left(\sum_{i=1}^3 r(c+i) - \text{copies} \right),$$

where $c \geq 3$ and

$$\mathcal{N}_c M(G_{(r,s,4)}) \cong Z_d \oplus \dots \oplus Z_d \quad \left(\sum_{i=1}^4 r(c+i) - \text{copies} \right),$$

where $c \geq 4$.

Remark 3.6. The above theorem determines the complete structure of n^{th} -nilpotent product of two cyclic groups, when $n \leq 4$. These results generalize Gupta and Moghaddam [3].

Finally, we remark that if one can construct a similar identity congruence as in Lemma 2.1, one may obtain the same results for $n \geq 5$. This certainly involves a very complicated commutators manipulations.

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